Median Filter Based Realizations of the Robust Time-Frequency Distributions

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Abstract — Recently, some new efficient tools for spectral analysis of signals corrupted by impulse noise have been introduced. They are based on the minimax robust M-estimation. The proposed robust distributions are calculated through an iterative procedure. Here we propose the robust short-time Fourier transform (STFT) and the robust Wigner distribution (WD), based on the simple median filter. Their efficiency in time-frequency analysis is demonstrated in examples.

I. INTRODUCTION

In many practical applications, especially in communications, signals are disturbed by a kind of impulse noise. These noises are commonly modeled by heavy-tailed probability density functions (pdf) [1]-[4]. It is a well known fact that the conventional time-frequency (TF) distributions [5], [6] are quite sensitive to this kind of noise, which is able to destroy sensitive signal information. It has been shown that the minimax Huber estimates can be used in order to design the periodogram, robust with respect to the impulse noise (the robust M-periodogram) [7], [8]. In particular, it has been shown that the robust M-periodogram, with the absolute error loss function, provided the maximum likelihood (ML) estimate of the spectrum for signals corrupted by the Laplacian noise. This periodogram can be successfully used for a wide class of impulse noises with other pdfs. For nonstationary frequency modulated (FM) signals corrupted by an impulse noise, the robust spectrogram (SPEC) [9] is introduced in analogy with the robust M-periodogram [7]. A robust version of the WD is defined in [10]. In this paper, we propose simple and efficient robust TF representations whose realizations are based on the median filters.

II. THEORETICAL BACKGROUND

Consider a real-valued signal \( x(n) \) corrupted by a white noise, \( y(n) = x(n) + \varepsilon(n) \). A general filtering task can be understood as finding a signal \( \hat{x}(n) \), as close as possible to \( x(n) \), by using finite number of samples of \( y(n) \). Signal \( \hat{x}(n) \) can be obtained by using \( 2M + 1 \) samples around instant \( n \), by minimizing

\[
J(m; n) = \sum_{k=n-M}^{n+M} F(e) = \sum_{k=n-M}^{n+M} F(y(k) - m), \quad (1)
\]

where \( F(e) \) is the loss function, while \( e = e(n, k) = y(k) - m \) is the error function. Minimum of (1) can be determined by solving the equation

\[
\frac{\partial J(m; n)}{\partial m}|_{m=\hat{x}(n)} = 0. \quad (2)
\]

Widely known moving average filter is the solution of (2) for squared error loss function \( F(e) = |e|^2 \)

\[
\hat{x}(n) = \frac{1}{2M + 1} \sum_{k=n-M}^{n+M} y(k)
\]

while median filter is the solution of (1) for \( F(e) = |e| \):

\[
\hat{x}(n) = \text{median}\{y(k), k \in [n-M,n+M]\}. \quad (3)
\]

Although these two filters are produced by similar loss functions, they exhibit quite different sensitivity to the noise influence. The moving average is an ML signal estimate in the case of Gaussian white noise. For impulse noise it produces poor results. The median filter is an ML estimate for the Laplacian white noise, while for the Gaussian noise it produces slightly worse results than the moving average. In addition, the median filter works well
in other kinds of impulse noise environments. In general, the minimax Huber’s estimation theory provides concepts for the loss function selection for various heavy-tailed noise distributions [2], [3].

III. The Robust Short-Time Fourier Transform

Consider noisy samples of a signal $x(t)$, $y(t+nT) = x(t+nT) + \varepsilon(t+nT)$, where $T$ is the sampling interval. Assume that the signal $x(t)$ and the noise $\varepsilon(t)$ are complex-valued. The problem is to find the STFT of $x(t)$ from noisy observations. By using the previous analysis the STFT can be defined as

$$STFT(t, \omega) = \arg\min_m \sum_{n=-N/2}^{N/2-1} F(e(t, \omega, n))$$

(5)

$$e(t, \omega, n) = y(t+nT)e^{-j\omega nT} - m.$$  

(6)

The standard STFT is a solution of (5) for the loss function $F(e) = |e|^2$

$$STFT_y(t, \omega) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} y(t+nT)e^{-j\omega nT}$$

= mean \{ y(t+nT)e^{-j\omega nT}, n \in [-N/2, N/2] \}.

(7)

Unfortunately, for other loss functions there is a problem in determining the minimum of (5), due to complex nature of $e(t, \omega, n)$. An iterative procedure for solving Eq. (5), with the loss function $F(e) = |e|$, is proposed in [7], [8]. The iterative procedure has to be done for each point in time-frequency plane, what makes it computationally very consuming. This solution is called robust periodogram [7], [8], [9]. We will call “robust” all solutions produced by the loss function $F(e) = |e|$, as it is common in the literature treating this kind of problems [2], [4]. Next we will introduce the median based formulation of the robust time-frequency representations.

Minimization of (5) can be understood as filtering of the vector-valued variable $y(t+nT)e^{-j\omega nT}$, where real and imaginary parts of $y(t+nT)e^{-j\omega nT}$ are vector coordinates. For this purpose we present two solutions.

1) The vector median (VM), proposed in [11], can be used to get a robust estimate of the STFT. Define $STFT_{VM}(t, \omega) = m$ where $m \in \{y(t+kT)e^{-j\omega kT}, k \in [-N/2, N/2]\}$, and for all $k \in [-N/2, N/2]$ the following inequality holds

$$\sum_{n=-N/2}^{N/2-1} |m - y(t+nT)e^{-j\omega nT}|$$

$$\leq \sum_{n=-N/2}^{N/2-1} |y(t+kT)e^{-j\omega kT} - y(t+nT)e^{-j\omega nT}|,$$

(8)

i.e.,

$$(\text{Re}\{STFT_{VM}(t, \omega)\}, \text{Im}\{STFT_{VM}(t, \omega)\}) = VM\{y(t+nT)e^{-j\omega nT}, n \in [-N/2, N/2]\}.$$  

(9)

The corresponding robust SPEC is $SPEC_{VM}(t, \omega) = \text{Re}\{STFT_{VM}(t, \omega)\}^2 + \text{Im}\{STFT_{VM}(t, \omega)\}^2$.

2) The marginal median (MM) gives simpler solution

$$STFT_{MM}(t, \omega) =$$

$$\text{median}\{\text{Re}\{y(t+nT)e^{-j\omega nT}\}, n \in [-N/2, N/2]\}$$

$$+j\text{median}\{\text{Im}\{y(t+nT)e^{-j\omega nT}\}, n \in [-N/2, N/2]\}.$$  

(10)

The vector median treats $(\text{Re}\{STFT_{VM}(t, \omega)\}, \text{Im}\{STFT_{VM}(t, \omega)\})$ as a vector whose components are mutually dependent, as opposed to the marginal median. After an intensive research and statistical comparison of the vector and marginal medians, in [12], “It has been found that the vector median filters and marginal median filters have similar performances despite the fact that they have different definitions. If we take into account the fact that the vector median filter has higher computational complexity, we can conclude that marginal median filter is preferable in most practical systems.” From our experiments we have found that a similar situation takes place for the proposed robust spectra, i.e., that the results obtained by using
the previous two algorithms are very close. In Fig.1, we have given an illustrative comparison of these three possible ways to estimate the solution of the minimization problem (5). We used a sinusoidal signal with impulse noise, and calculated the STFT at one point in the TF plane. All possible values of $y(t + nT)e^{-j2\omega nT}$ are given on the left-hand side of Fig.1. Zoomed region where the exact solution lies is given on the right-hand side. The exact solution for this point was $STFT(t, \omega) = 1 + j0$. Note that the iterative procedure [7] (marked by *') produced the value 0.9659 + 0.0641 within 5 iterations. The vector median (marked by +) resulted in the value 0.1781 − j0.0006 while the marginal median (marked by o) resulted in 1.0462 + j0. We can see that all three methods gave solutions close to each other. The signal length in the considered example was $N = 1024$, while the signal-to-noise ratio was $SNR = -14.77$ dB. Note that the presented realization is a typical result that we obtained in numerous experiments. The accuracy was always of the same order like in this example.

IV. The Robust Wigner and Other Quadratic Distributions

Previous analysis can be extended to other TF representations. We can introduce the $M$-estimate of the WD as a solution of the problem

$$WD_y(t, \omega) = \arg\min_{n \in [-N/2, N/2]} \sum_{n=-N/2}^{N/2} F(e(t, \omega, n)), \quad (11)$$

$$e(t, \omega, n) = \text{Re}\{y(t+nT)y^*(t-nT)e^{-j2\omega nT}\} - m. \quad (12)$$

The quadratic loss function $F(e) = |e|^2$ gives the conventional WD definition

$$WD_y(t, \omega) =$$

$$\frac{1}{N+1} \sum_{n=-N/2}^{N/2} y(t + nT)y^*(t - nT)e^{-j2\omega nT}$$

$$= \text{mean}\{y(t+nT)y^*(t-nT)e^{-j2\omega nT},$$

$$n \in [-N/2, N/2]\}$$

$$= \text{mean}\{\text{Re}\{y(t+nT)y^*(t-nT)e^{-j2\omega nT}\},$$

$$n \in [-N/2, N/2]\}.$$ 

It can be easily concluded from (11) that for an even and real loss function the general WD remains real. It means that for loss function $F(e) = |e|$ the robust WD is equal to

$$WD_R(t, \omega) = \text{median}\{\text{Re}\{y(t+nT)y^*(t-nT)e^{-j2\omega nT}\},$$

$$n \in [-N/2, N/2]\}.$$ \quad (14)

Generally, it can be shown that any robust TF distribution, obtained by using the Hermitian local autocorrelation function (LAF), $r_{yy}(t, nT) = r_{yy}^*(t, -nT)$ in the minimization, is real for even loss function. In the WD case this condition is satisfied, since $r_{yy}(t, nT) = y(t + nT)y^*(t - nT).$ For a general quadratic distribution from the Cohen class [6], with a Hermitian LAF, the proposed robust version reads

$$CD_R(t, \omega) = \text{median}\{\text{Re}\{r_{yy}(t, nT)e^{-j2\omega nT}\},$$

$$n \in [-N/2, N/2]\}, \quad (15)$$

where $r_{yy}(t, nT)$ includes the kernel in time-lag domain.

A. Examples

In the examples, the robust distribution is calculated by using the loss function $F(e) = |e|$ and the marginal median.

Example 1: Consider signal corrupted with a high amount of heavy-tailed noise:

$$x(t) = e^{j\omega_1 t} + 2(e_{11}^2(t) + j e_{12}^2(t))$$ \quad (16)$$

where $e_1(t)$ and $e_2(t)$ are mutually independent white Gaussian noises $N(0, 1)$, and $\omega_1 = 256\pi$. The signal is sampled with $T = 1/512$. Window length is $N = 128$. The standard and the robust SPECs are shown in Figs.2a,b, respectively.

Example 2: Next, consider two signals with time-varying instantaneous frequencies

$$x(t) = e^{j\omega_2(t+0.5)^2 - j\omega_2(t+0.5)}e^{-4(t-0.6)^2}$$

$$+ e^{j\omega_2(t+0.5)^2 + j\omega_2(t+0.5)}e^{-4(t+0.6)^2}$$

$$+ 0.5(e_{21}^2(t) + j e_{22}^2(t))$$ \quad (17)$$
$x(t) = e^{ja_4 \cos(3\pi t/2)} + 0.5(e_1^2(t) + je_2^2(t))$ (18)

where $a_2 = 102.4\pi$, $a_3 = 64\pi$ and $a_4 = 256/3$. The signals are sampled with $T = 1/1024$. Window width is $N = 256$. The standard and the robust WDs of these signals are shown in Figs.2c-f. For the graphical presentation, the standard WDs are limited to the value of the WD maximum for nonnoisy signal.

From these examples we can observe a significant improvement of the TF representation by using the robust distributions instead of the standard ones, in the case of impulse noise.

**Example 3**: For statistical analysis of the standard and the robust WD, as instantaneous frequency (IF) estimators,

$$\hat{\omega}(t) = \arg \max_\omega WD(t, \omega)$$ (19)

we have considered the signal

$$x(t) = e^{ja_{3t^2}} + \varepsilon(t)$$ (20)

corrupted by three kinds of noise: a) Gaussian white, complex noise, $\varepsilon(t) = 1.5a(\varepsilon_1(t) + je_2(t))$; b) Cube of the Gaussian noise $\varepsilon(t) = a(\varepsilon_1^3(t) + je_2^3(t))$; and c) Complex, white, Cauchy noise $\varepsilon(t) = \varepsilon_3(t; a) + je_4(t; a)$, where $\varepsilon_3(t; a)$ and $\varepsilon_4(t; a)$ are mutually independent with the pdf $p(x) = a/\pi(1 + (ax)^2)$, while $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are the same as in the previous examples. We have considered various amounts of noise, being defined by $a \in [0, 1]$, and performed 100 runs for both WDs and various $a$. Parameter $a$ is used to describe the amount of noise, since the common signal to noise ratio is not an appropriate measure for impulse noise environments. Considered time interval was $t \in [-0.5, 0.5]$ with $T = 1/1024$. The mean squared error of the IF estimation, $E\{(\omega(t) - \hat{\omega}(t))^2\}$, is shown in Fig.3. Here, $\omega(t)$ is the true IF value $\omega(t) = 2a_3t$. From this figure it can be concluded that: 1) For the pure Gaussian noise the standard WD produces slightly better results than the robust WD, Fig.3a; 2) Even a small amount of an impulse noise significantly degrades estimation performance of the standard WD, while the proposed robust WD is far less sensitive to this kind of noise, Fig.3b, 3c.

V. Conclusion

Simple, median based, formulation of the robust TF distributions is presented. Special attention has been paid to the robust spectrogram and the robust Wigner distribution. The marginal median, as numerically the most efficient robust form, has been used for the robust spectrogram realization. Since the robust Wigner distribution is real-valued, its realization can be performed by using standard median form. Simulation results confirm favor-
Fig. 2. Time-frequency representations of signals with heavy-tailed noise by using: a) Standard spectrogram, b) Robust spectrogram, c) Standard Wigner distribution, d) Robust Wigner distribution, e) Standard Wigner distribution, f) Robust Wigner distribution.

Fig. 3. Mean squared error of the instantaneous frequency estimation for: a) Signal corrupted by the Gaussian noise, b) Signal corrupted by the heavy-tailed noise (cube of Gaussian noise), c) Signal corrupted by the Cauchy noise. Dotted line - by using the standard WD; solid line - by using the robust WD. Parameter $\alpha$ indicates the amount of noise.
able properties of robust distributions when the noise is of impulse nature. The instantaneous frequency estimators based on the robust and the standard Wigner distribution are compared in various noisy environments. It has been shown that the robust Wigner distribution outperforms the standard distribution in impulse noise environments.

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References